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**Abstract:** We study in this report the $H_\infty$-stabilizing properties of PI controllers for particular SISO delay systems of three classes : dead-time systems as well as retarded or neutral systems. For a subclass of neutral systems, this enables us to deduce a parametrization of all $H_\infty$-stabilizing controllers.

**Key-words:** delay system, $H_\infty$-stability, PI controller, coprime factorization, parametrization

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# Stabilization d'une classe de systèmes à retards au moyen de contrôleurs PI

**Résumé :** Dans ce rapport, nous étudions les propriétés des contrôleurs de type PI pour la stabilisation $H_\infty$ de certains systèmes à retards SISO appartenant à trois classes : la classe des systèmes dont le transfert est du type $R(s)e^{-sT}$ où $R$ est rationnel, la classe des sytèmes retardés et la classe des systèmes neutres. Pour une sous-classe de systèmes neutres, ces resultats nous permettent de déduire une paramétrisation de l'ensemble des contrôleurs stabilisants.

**Mots-clés :** système à retards, stabilité $H_\infty$, contrôleur PI, factorisation copremière, paramétrisation

# 1 Introduction

We consider in this report the $H_\infty$-stabilization, in a standard feedback context, of various types of SISO delay systems. We shall be concerned here with the analysis of two methods that have been used extensively in the feedback stabilization of delay systems, namely, PI controllers and coprime factorization controllers. The first is conceptually more simple, as it requires the calculation of two parameters to define the controller; the second needs more algebraic methods, since the parametrization of stabilizing controllers is commonly expressed in terms of solutions to a Bézout equation over the algebra $H^\infty(\mathbb{C}_+)$ of bounded analytic transfer functions defined on the right half-plane $\mathbb{C}_+$.

An important recent reference for the PID stabilization of first order dead-time systems is [11]. This gives the complete set of stabilizing PID parameters for both open-loop stable and unstable plants. Moreover in the case of an open-loop unstable plant, it gives a necessary and sufficient condition on the time delay for the existence of stabilizing PID controllers.
We shall analyse here similar questions, but with tools that seem to be simpler than those of [11] which are related to the Hermite–Biehler theorem. Moreover we will examine a different class of dead-time systems as well as some retarded and neutral systems for the first time. Note that the $H_\infty$-stability of a neutral delay system is a more restrictive notion than the absence of unstable poles [9].
Another reference on this question is the book of Niculescu. For questions of feedback stability, when considering the PI-stabilization of a system with transfer function $G(s) = \dfrac{e^{-sh}}{s}$, he is mainly concerned with stability of the closed-loop transfer function $G(s)K(s)(1+G(s)K(s))^{-1}$, where $K$ denotes a controller; we shall analyse the a priori stronger notion of input-output stabilizability (the four standard transfer functions have to be in $H_\infty$), although our conclusions are similar in the special cases analysed in [7].
We mention also [4, 12, 13] as papers that consider PID control of delay systems in a robustness or $H_\infty$ framework, although their approaches are totally different from ours and their results only loosely related.

We consider in section 2 dead-time systems of the type $G(s) = e^{-sT}\dfrac{p(s)}{q(s)}$ and give some results on the existence of PI stabilizing controllers. Also, in the case where $\deg p$ and $\deg q$ are less than or equal to one, the existence of stability windows is examined.

Retarded and neutral systems of the type $G(s) = \dfrac{1}{p(s) + q(s)e^{-sT}}$ are then analysed in section 3. Many engineering or biological models leading, possibly after linearization, to such transfer functions can be found in ([7], chap 2). We give here some conditions (necessary and sufficient in the case of a neutral system and sufficient otherwise) for the existence of PI stabilizing controllers and use these results to determine a parametrization of all stabilizing controllers in the particular case where $\deg p = \deg q = 1$.

## 2 Dead-time systems

We consider in this section delay systems with transfer function

$$G(s) = e^{-sT}\frac{p(s)}{q(s)}, \tag{1}$$

where $T > 0$ and $p$ and $q$ are real polynomials satisfying $\deg p \leq \deg q$ in order to deal with proper systems (i.e. ones such that $\sup_{\mathrm{Re}\, s>0,\ |s|\geq R} |G(s)| < \infty$, for some $R > 0$).
We want to evaluate for such systems the $H_\infty$-stabilizing properties of PI controllers, that is, of controllers with transfer function

$$K(s) = k_p + \frac{k_i}{s}, \tag{2}$$

where $k_p$ and $k_i$ are real coefficients.

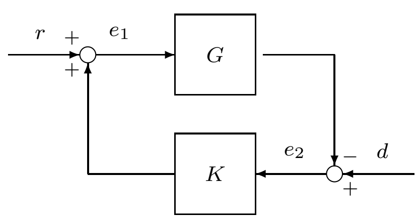


Figure 1: Standard Feedback Configuration

We consider the standard feedback scheme of Figure 1, and recall that $H_\infty$-stability is equivalent to the three transfer functions $(1 + KG)^{-1}$, $K(1 + GK)^{-1}$, $G(1 + KG)^{-1}$ being stable.

We have

$$\frac{sq(s)}{sq(s)+e^{-sT}p(s)(k_p s+k_i)}, \frac{sp(s)e^{-sT}}{sq(s)+e^{-sT}p(s)(k_p s+k_i)}, \frac{k_p sq(s)+k_i q(s)}{sq(s)+e^{-sT}p(s)(k_p s+k_i)},$$

and those transfer functions represent the behaviour of a retarded delay system if $\deg p < \deg q$ and of a neutral delay system if $\deg p = \deg q$.

We consider first the case of a neutral-type closed loop; the next proposition, without giving a complete characterization of the values of $k_p$ and $k_i$ solving the problem, allows one to fix the location of $k_p$ and to determine, for a fixed $k_p$, what is obtainable.

**Proposition 2.1** *Consider the system* $G(s) = e^{-sT}\dfrac{p(s)}{q(s)}$ *where* $\deg p = \deg q$, $T > 0$ *and the controller* $K(s) = k_p + \dfrac{k_i}{s}$.
*1) If* $|\lim_{|s|\to\infty} \frac{q(s)}{p(s)k_p}| \leq 1$ *then* $K(s)$ *does not stabilize* $G(s)$.
*2) If* $|\lim_{|s|\to\infty} \frac{q(s)}{p(s)k_p}| > 1$ *then the transfer functions of the closed loop have at most finitely many unstable poles and for a fixed* $k_p$*, if there exists a* $k_i$ *such that there is no unstable pole then* $K(s)$ *stabilizes* $G(s)$.

**Proof** 1) From Proposition 2.1 in [9], we have that if $|\lim_{|s|\to\infty} \frac{q(s)}{p(s)k_p}| < 1$ then $K(s)$ does not stabilize $G(s)$. The equality part follows from Theorem 2.1 in [9].
2) From Proposition 2.1 in [9], we have that if $|\lim_{|s|\to\infty} \frac{q(s)}{p(s)k_p}| > 1$ then the transfer functions of the closed loop have at most finitely many unstable poles and if they have none then they lie in $H_\infty$. So that, the coefficient $k_p$ being fixed, if there exists a $k_i$ giving poles only in the left half-plane then the closed loop is stable.

**Remark 2.1** *In ([7], page 124) the same observation as in 1) is made in the particular case of* $p$ *and* $q$ *of degree one.*

We examine now the natural question of stabilizing $G$ with a PI controller which has been designed to stabilize $G$ when $T = 0$. The next proposition partially answers this question. First we recall the standard fact that

**Remark 2.2** *A quadratic equation* $ax^2 + bx + c$ *with* $a > 0$ *and* $b$, $c$ *real has*

1. *one positive root and one negative root if* $c < 0$;
2. *two real roots if* $b^2 - 4ac > 0$; *and*

*3. two non-real roots if* $b^2 - 4ac < 0$.

**Proposition 2.2** *Suppose the PI controller with transfer function* $K(s) = k_p + \dfrac{k_i}{s}$ *stabilizes* $\dfrac{p(s)}{q(s)}$ *(that is the system (1) where* $T = 0$*). Then,*
*1) If* $\deg p < \deg q$, $K$ *stabilizes* $\dfrac{p(s)}{q(s)}e^{-sT}$ *for sufficiently small* $T$.
*2) If* $\deg p = \deg q$ *and* $|\lim_{|s|\to\infty} \dfrac{q(s)}{p(s)k_p}| > 1$, $K$ *stabilizes* $\dfrac{p(s)}{q(s)}e^{-sT}$ *for sufficiently small* $T$.
*3) In the particular case of systems of the type* $G(s) = \dfrac{e^{-sT}}{s-\sigma}$, *the first value* $T_1$ *of the delay which will destabilize the closed loop satisfies*

$$\cos(\omega T_1) = \frac{k_i \omega^2 + k_p \sigma \omega^2}{k_i^2 + k_p^2 \omega^2} \qquad \textit{and} \qquad \sin(\omega T_1) = \frac{\omega^3 k_p - \sigma \omega k_i}{k_i^2 + k_p^2 \omega^2},$$

*where* $\omega = \sqrt{\frac{1}{2}\left(k_p^2 - \sigma^2 + \sqrt{(\sigma^2 - k_p^2)^2 + 4k_i^2}\right)}$.
*For* $T > T_1$, *the closed loop is unstable, that is there is no 'stability windows' phenomenon.*
*4) In the particular case of systems of the type* $G(s) = \dfrac{s-\alpha}{s-\beta}e^{-sT}$, $K(s)$ *will have the following properties :*
*- if* $|k_p| < 1$ *than* $K$ *stabilizes* $\dfrac{p(s)}{q(s)}e^{-sT}$ *for sufficiently small* $T$ *and the first value* $T_1$ *which destabilizes the system is such that the closed loop* $[G, K]$ *remains unstable for all* $T > T_1$. *That is, here again there is no 'stability windows' phenomenon,*
*- if* $|k_p| > 1$, *the closed loop* $[G, K]$ *is unstable for all* $T > 0$.

**Remark 2.3** *The proof of Proposition 2.2 relies extensively on the Walton–Marshall techniques (see [5, 8]) which we briefly recall here.*
*This method allows one to discover whether all the roots of* $A(s) + C(s)e^{-sT}$, *where* $A$ *and* $C$ *are real polynomials, lie in the left-half plane and to determine the range(s) of values of* $h$ *for which this is so.*
*We recall that it consists of three steps. The first step is to examine stability for* $T = 0$. *The second step considers the case of infinitesimally small positive* $T$ *and tries to find whereabouts in the complex plane the infinite number of new roots have appeared. The third step is to find positive values of* $h$ *if any at which there are zeros of* $A(s) + C(s)e^{-sh}$ *lying on the imaginary axis and to determine whether the zeros touch the axis or cross from one half-plane to the other with increasing* $h$. *Walton and Marshall made the observation that if* $A(i\omega) + C(i\omega)e^{-i\omega h}$ *has a zero at* $\omega_0$ *then* $W(\omega_0^2) := A(i\omega_0)A(-i\omega_0) - C(i\omega_0)C(-i\omega_0) = 0$.

*Proof of Proposition 2.2:* 1) and 2) follow easily from [6] : if $K$ is such that $|\frac{p(\infty)}{q(\infty)}K(\infty)| < 1$ then $[\frac{p}{q}, K]$ is w-stable; that is, there exists $T_0$ such that $[\frac{p}{q}e^{-sT}, K]$ is stable for all $T \in (0, T_0)$.
Note that when $\deg p < \deg q$ the above condition is always satisfied.
3) Let $A(s) = sq(s) = s(s-\sigma)$ and $C(s) = p(s)(k_p s + k_i) = k_p s + k_i$. We form

$$W(\omega^2) = A(i\omega)A(-i\omega) - C(i\omega)C(-i\omega) = \omega^2 q(i\omega)q(-i\omega) - (k_p^2\omega^2 + k_i^2)p(i\omega)p(-i\omega). \quad (3)$$

The denominator of each closed-loop transfer function is

$$sq(s) + e^{-sT}p(s)(k_p s + k_i) = s(s-\sigma) + e^{-sT}(k_p s + k_i).$$

As the closed loop is stable at $T = 0$, we obtain from the Routh criterion that $k_p > \sigma$ and $k_i > 0$.
We have that $W(\omega^2) = \omega^4 + (\sigma^2 - k_p^2)\omega^2 - k_i^2$, which has a unique positive zero, given by

$$\omega = \sqrt{\frac{1}{2}\left(k_p^2 - \sigma^2 + \sqrt{(\sigma^2 - k_p^2)^2 + 4k_i^2}\right)}. \quad (4)$$

We can then conclude that the first delay $T_1$ which will destabilize the closed loop will be defined by

$$\cos(\omega T_1) = \mathrm{Re}\{-\frac{i\omega(i\omega - \sigma)}{k_p i\omega + k_i}\} = \frac{k_i\omega^2 + k_p\sigma\omega^2}{k_i^2 + k_p^2\omega^2}$$

and

$$\sin(\omega T_1) = \mathrm{Im}\{\frac{i\omega(i\omega - \sigma)}{k_p i\omega + k_i}\} = \frac{\omega^3 k_p - \sigma\omega k_i}{k_i^2 + k_p^2\omega^2} \quad (5)$$

and, the root $\omega$ being unique, we have that for $T > T_1$ the closed loop will be unstable. This is because (see [5] page 25) if $W(\omega^2) = 0$ has no repeated roots the stabilizing and destabilizing roots alternate and as $W(\omega^2) > 0$ for large $\omega$ the highest root is always destabilizing. Indeed, there is no 'stability windows' phenomenon.
4) In this case, $A(s) = s(s-\beta)$ and $C(s) = (s+\alpha)(k_p s + k_i)$ so that

$$W(\omega^2) = (1 - k_p^2)\omega^4 + (\beta^2 - \alpha^2 k_p^2 - k_i^2)\omega^2 - \alpha^2 k_i^2. \quad (6)$$

Now, the denominator of the closed loop is equal to

$$(1 + k_p)s^2 + (\beta + k_i + k_p\alpha)s + \alpha k_i$$

The closed loop being stable at $T = 0$, there exists $k_p$ and $k_i$ satisfying

$$\left\{\begin{array}{l} 1+k_p > 0 \\ \beta + k_i + k_p\alpha > 0 \\ \alpha k_i > 0 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} 1+k_p < 0 \\ \beta + k_i + k_p\alpha < 0 \\ \alpha k_i < 0 \end{array}\right.$$

If $-1 < k_p < 1$ then $1 - k_p^2 > 0$ and $\dfrac{1}{k_p} > 1$ so that $W(\omega^2) > 0$ for large $\omega$ and the modulus of the coefficient of the highest degree term of $A(s)$ is greater than the modulus of the coefficient of the highest degree term of $C(s)$ which implies that $[G, K]$ is stable for sufficiently small $T$ (see ([5], page 25)).
Now from Remark 2.2, we can conclude that equation (6) has only one positive root and for the same reason as above this unique root is destabilizing and for $T > T_1$ the closed loop will be unstable (here again there cannot be any 'window phenomenon' here).
If $k_p < -1$ or $k_p > 1$ then $W(\omega^2) < 0$ for large $\omega$ and $[G, K]$ is unstable for all $T$. ■

**Remark 2.4** *(i) An extension of part 3 to systems $G(s) = e^{-sT}\lambda\frac{s-\alpha}{s-\beta}$ and of part 4 to systems $G(s) = \frac{\lambda e^{-sT}}{s-\sigma}$ with $\lambda > 0$ can easily be deduced by replacing $G(s)$ with $\lambda^{-1}G(s)$ and $K(s)$ by $\lambda K(s)$, which does not change the internal stability properties of the closed loop.*
*(ii) The case $\sigma = 0$ of 1) above is treated in [7, Sec. 4.6]*
*(iii) If $G(s) = \dfrac{e^{-sT}}{q(s)}$ with $\deg q(s) = 2$, we can have stability window phenomena (see Proposition 7.15 and Remark 7.10 of [7]).*
*(iv) Part 3 of Proposition 2.2 allows one to find PI controllers that are robust relative to coprime factor perturbations.*

# 3 Neutral and retarded delay systems

We shall consider in this section systems with transfer function

$$G(s) = \frac{1}{p(s) + q(s)e^{-sT}},$$

with $T > 0$ and $\deg p \geq \deg q$. If $\deg p = \deg q$ the system is of neutral type, otherwise it is of retarded type.

Again we take $K(s) = k_p + k_i/s$; then the three closed-loop transfer functions are given by

$$\begin{aligned}
(I + K(s)G(s))^{-1} &= \frac{s(p(s) + q(s)e^{-sT})}{s(p(s) + q(s)e^{-sT}) + (k_i + k_p s)}, \\
G(s)(I + K(s)G(s))^{-1} &= \frac{sr(s)}{s(p(s) + q(s)e^{-sT}) + (k_i + k_p s)}, \\
K(s)(I + G(s)K(s))^{-1} &= \frac{(p(s) + q(s)e^{-sT})(k_i + k_p s)}{s(p(s) + q(s)e^{-sT}) + (k_i + k_p s)}.
\end{aligned}$$

When considering the stability of $G$ by a controller $K$ which has been designed for the undelayed system, it is evident from the Walton–Marshall results that the stability properties of $(G, K)$ are controlled by the real positive values of $\omega^2$ (if any) that satisfy

$$W(\omega^2) := A(i\omega)A(-i\omega) - C(i\omega)C(-i\omega) = 0,$$

where $A(s) = sp(s) + (k_i + k_p s)$ and $C(s) = sq(s)$.

**Proposition 3.1** *Let* $G(s) = \dfrac{1}{p(s) + q(s)e^{-sT}}$ *with real coefficients, and* $K(s) = k_p + k_i/s$ *with* $k_p,\ k_i \in \mathbb{R}$.

*(i) If* $\deg p = \deg q$ *and* $|\lim_{|s|\to\infty} p(s)/q(s)| \leq 1$ *then there is no finite-dimensional controller stabilizing* $G$ *and hence* $[G, K]$ *is not* $H_\infty$*-stable.*

*(ii) If* $\deg p = \deg q \geq 1$ *and* $|\lim_{|s|\to\infty} p(s)/q(s)| > 1$ *then every PI controller* $K$ *which stabilizes* $G$ *when* $T = 0$ *will also stabilize* $G$ *when* $T$ *is sufficiently small.*

*Moreover, in the particular case where* $p(s) = \alpha s + \beta$ *and* $q(s) = \gamma s + \delta$, *if* $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2(\alpha^2 - \gamma^2) < 0$ *then* $K$ *will stabilize* $G$ *for all* $T$.

*(iii) If* $p(s) = \beta$ *and* $q(s) = \delta$ *for all* $s$, *and* $|(\beta + k_p)/\delta| > 1$, *then every PI controller* $K$ *which stabilizes* $G$ *when* $T = 0$ *will also stabilize* $G$ *for all* $T$. *If* $|(\beta + k_p)/\delta| < 1$ *then* $[G, K]$ *is unstable for all* $T > 0$.

*(iv) If* $\deg p > \deg q$ *then every PI controller* $K$ *which stabilizes* $G$ *when* $T = 0$ *will also stabilize* $G$ *when* $T$ *is sufficiently small.*

*Moreover, in the particular case where* $p(s) = \alpha s + \beta$ *and* $q(s) = \delta$, *if* $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2\alpha^2 < 0$ *then* $K$ *will stabilize* $G$ *for all* $T$.

**Proof** (i) This is a special case of Theorem 4.1 in [9].

(ii) Suppose $K$ stabilizes $G$ at $T = 0$. Let $\alpha_0$ and $\gamma_0$ be the coefficients of the terms of highest degree of $p(s)$ and $q(s)$ respectively. Then, the coefficient of the term of highest degree of $W(\omega^2)$ is equal to $(\alpha_0^2 - \gamma_0^2)$ so that $W(\omega^2)$ is positive for large $\omega$. Indeed we also have that the coefficient of the highest degree term in $A$, $\alpha_0$, is of modulus greater than the coefficient of the highest degree term in $C$, $\gamma_0$, and we can deduce from ([5], page 25) that $[G, K]$ is stable for sufficiently small $T$. In the particular case where $p(s) = \alpha s + \beta$ and $q(s) = \gamma s + \delta$, a straightforward calculation gives

$$W(\omega^2) = (\alpha^2 - \gamma^2)\omega^4 + (-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)\omega^2 + k_i^2.$$

If $|\lim_{|s|\to\infty} p(s)/q(s)| > 1$ , we know from Proposition 2.1 in [9] that if the 3 closed-loop transfer functions have no poles in the right-half plane then $[G, K]$ is $H_\infty$-stable.
Now, if $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2(\alpha^2 - \gamma^2) < 0$, there is no positive solution for $W(\omega^2) = 0$ (see Remark 2.2) and the closed loop is stable for all $T$.
(iii) In this case $A(s) = (\beta + k_p)s + k_i$, $C(s) = \delta s$, $W(\omega^2) = ((\beta + k_p)^2 - \delta^2)\omega^2 + k_i^2$ and the result follows as in ii).
(iv) As $H_\infty$-stability of retarded systems is equivalent to the condition 'no poles in the right half-plane', the Walton-Marshall techniques imply the result. First, the infinite number of new roots appearing when $T$ is not zero are located in the left half-plane. Second, in the particular case of $p$ and $q$ of degree one, it is easy to see that if $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2\alpha^2 < 0$ there is no positive solution for $W(\omega^2) = 0$ and the closed loop remains stable for all $T$. // Finally, we apply the above results in order to provide a characterization of the set of stable controllers in some important examples without needing to determine the unstable poles explicitly (in general they would be given as the solutions of a transcendental equation).

**Theorem 3.1** *Let* $G(s) = \dfrac{1}{\alpha s + \beta + (\gamma s + \delta)e^{-sT}}$ *with* $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{R}$. *Suppose that* $|\alpha| > |\gamma|$; *then the set of all* $H_\infty$*-stabilizing controllers is given by* $\dfrac{V + MQ}{U - NQ}$ *where* $N(s) = \dfrac{1}{s+1}$,
$M(s) = \dfrac{(\alpha s + \beta) + (\gamma s + \delta)e^{-sT}}{s+1}$,
$U(s) = \dfrac{s(s+1)}{((\alpha s + \beta) + (\gamma s + \delta)e^{-sT})s + k_p s + k_i}$, $V(s) = \dfrac{(s+1)(k_i + k_p s)}{((\alpha s + \beta) + (\gamma s + \delta)e^{-sT})s + k_p s + k_i}$,
$Q$ *is a free parameter in* $H_\infty$ *and* $k_p$, $k_i$ *satisfy the conditions* $k_p + \beta + \delta > 0$, $k_i > 0$ *and* $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2(\alpha^2 - \gamma^2) < 0$.

**Proof** Suppose first that $\gamma \neq 0$ and $\left|\frac{\alpha}{\gamma}\right| > 1$.

If $\alpha > 0$, as $|\frac{\alpha}{\gamma}| > 1$ we have that $\alpha + \gamma > 0$ and from above we can conclude that there exists a PI controller $k_p + \frac{k_i}{s}$ which stabilizes $G(s)$ when $T = 0$, provided that $k_p$ and $k_i$ satisfy the conditions $k_p + \beta + \delta > 0$ and $k_i > 0$.
Now, taking $k_p = -\beta + \sqrt{\delta^2 + 2\alpha k_i}$ (which satisfies $k_p + \beta + \delta > 0$) we have that $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2(\alpha^2 - \gamma^2) = -4k_i^2(\alpha^2 - \gamma^2) < 0$ and so there always exists a PI controller $K_0$ which stabilizes $G(s)$ for all $T > 0$.
If $\alpha < 0$, we have $\alpha + \gamma < 0$ and taking $k_i < 0$ and $k_p = -\beta - \sqrt{\delta^2 + 2\alpha k_i}$ allows us to obtain $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2(\alpha^2 - \gamma^2) < 0$.
In the case where $\gamma = 0$, similar calculations prove that there always exist $k_p$ and $k_i$ such that $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i\alpha^2 < 0$.
Now, as $G$ is $H_\infty$-stabilizable it necessarily admits a coprime factorization over $H_\infty$, that is, there exist $N$, $M$, $X$ and $Y \in H_\infty$ such that $G = \dfrac{N}{M}$ and $MX + NY = 1$. The set of all stabilizing controllers of $G$ is then given by the Youla parametrization $\dfrac{Y + MQ}{X - NQ}$, $Q \in H_\infty$ and there exists $Q_0 \in H_\infty$ such that $\frac{Y+MQ_0}{X-NQ_0} = \frac{V}{U} = K_0$. Note that $U$ and $V$ are Bézout factors associated to the coprime factorization $(N, M)$ as well.
It is easy to verify that $(N, M) = \left(\dfrac{1}{s+1}, \dfrac{(\alpha s + \beta) + (\gamma s + \delta)e^{-sT}}{s+1}\right)$ is a coprime factorization of $G$ over $H_\infty$ as $\inf_{\{\mathrm{Re}\, s > 0\}} (|N(s)| + |M(s)|) > 0$ and in [10] we can see that the calculation of Bézout factors becomes simpler once one already knows a stabilizing controller.
We have that $\dfrac{1}{1 + GK_0} = MU$ so that $U = \dfrac{s(s+1)}{((\alpha s + \beta) + (\gamma s + \delta)e^{-sT})s + k_p s + k_i}$; also $\dfrac{K_0}{1 + GK_0} = MV$ so that $V = \dfrac{(s+1)(k_i + k_p s)}{((\alpha s + \beta) + (\gamma s + \delta)e^{-sT})s + k_p s + k_i}$. We have $(-\delta^2 + (\beta + k_p)^2 - 2\alpha k_i)^2 - 4k_i^2(\alpha^2 - \gamma^2) < 0$ so that , as in the proof of Proposition 3.1, the Bézout factors $U$ and $V$ lie in $H_\infty$. The Youla parametrization of all stabilizing controllers of $G$ can now be given by $\dfrac{V + MQ}{U - NQ}$, $Q \in H_\infty$

**Remark 3.1** *In the case $\gamma = 0$, a parametrization of all $H_\infty$-stabilizing controllers has already been obtained through a direct calculation of a Bézout identity (see [1]). The advantage of the above result is to avoid a possible interpolation procedure.*

# 4 Conclusion

We have analysed the closed-loop stability of a wide collection of delay systems with respect to PI controllers. Further extensions of these ideas could be based on moe sophisticated

techniques for determining the positive roots of functions such as $W(\omega^2)$ above, and this would help us understand better the phenomenon of stability windows.
One remaining issue is to parametrize the stabilizing controllers of a delay system for which one does not have an exact knowledge of the unstable poles. (If these are known, then techniques such as those in [2] and [1] provide a solution.) Theorem 3.1 achieves this in some simple cases, by using the PI-based techniques developed in this paper, but the general case is clearly a topic for further research.

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